Newton's laws of motion in the form of a Riccati equation

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We discuss two applications of a Riccati equation to Newton's laws of motion. The first one is the motion of a particle under the influence of a power law central potential $V(r)=kr^{\epsilon}$. For zero total energy we show that the equation of motion can be cast in the Riccati form. We briefly show here an analogy to barotropic Friedmann-Robertson-Lemaitre cosmology where the expansion of the universe can be also shown to obey a Riccati equation. A second application in classical mechanics, where again the Riccati equation appears naturally, are problems involving quadratic friction. We use methods reminiscent to nonrelativistic supersymmetry to generalize and solve such problems.

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I. INTRODUCTION

It is known that Riccati equations, in general, of the type

$$\frac{dy}{dx}=f(x)y^2+g(x)y+h(x), \tag{1}$$

find surprisingly many applications in physics and mathematics. For example, supersymmetric quantum mechanics [1], variational calculus [2], nonlinear physics [3], renormalization group equations for running coupling constants in quantum field theories [4], and thermodynamics [5] are just a few topics where Riccati equations play a key role. The main reason for their ubiquity is that the change of function

$$y=-\frac{1}{f}\left[\frac{d}{dx}(\ln z)-\frac{g}{2}\right] \tag{2}$$

turns it into linear second-order differential equations of the form

$$\frac{d^2z}{dx^2}-\left(\frac{d}{dx}\ln f\right)\frac{dz}{dx}-\left[\frac{g^2}{4}-\frac{1}{2}\frac{dg}{dx}+h-\frac{d}{dx}\ln f\right]z=0 \tag{3}$$

that stand as basic mathematical background for many areas of physics.

Since the Riccati equation is a widely studied nonlinear equation, knowing that the physical system under consideration can be brought into Riccati form has certainly many advantages.

It is, therefore, of interest to look for yet different physical problems that are governed by this first-order nonlinear equation. This can be a starting point to new avenues in investigating analytical solutions of yet unsolved problems. In this paper we concentrate mainly on topics from classical mechanics and show that certain types of Newton's laws of motion are equivalent to the Riccati equation.

II. THE POWER LAW CENTRAL POTENTIALS

After implementation of the angular momentum conservation law, the equation for the energy conservation in the case of a central potential $V(r)$ is given by the standard expression

$$E=\frac{1}{2}m\dot{r}^2+\frac{l^2}{2mr^2}+V(r). \tag{4}$$

Taking a derivative with respect to time of Eq. (4) results into a second fundamental equation of the form

$$m\ddot{r}-\frac{l^2}{mr^3}+\frac{dV(r)}{dr}=0. \tag{5}$$

Specializing from now on to a power law potential [6]

$$V(r)=kr^{\epsilon}, \tag{6}$$

where k is the coupling constant and the exponent ϵ can be either positive or negative, we obtain from Eq. (5)

$$V(r)=-\frac{m\ddot{r}r}{\epsilon}+\frac{l^2}{\epsilon mr^2}. \tag{7}$$

Inserting the last equation in Eq. (4) gives

$$\frac{1}{2}m\dot{r}^2+\left(\frac{1}{2}+\frac{1}{\epsilon}\right)\frac{l}{mr^2}-\frac{m\ddot{r}r}{\epsilon}-E=0. \tag{8}$$

Under the assumption of $E=0$, this expression leads to a Riccati form. Obviously with $E=0$, we restrict ourselves to the case $k<0$. To explicitly derive from Eq. (8) the Riccati equation we pass (as it is costummary in central potential problems) to an angle θ as a free variable [i.e., we consider $r(\theta(t))$]. With

$$\dot{\theta}=\frac{l}{mr^2}, \quad r'\equiv\frac{dr}{d\theta}, \tag{9}$$

and introducing

$$\omega=\frac{r'}{r}, \tag{10}$$

it can be readily shown, after some algebraic manipulations, that Eq. (8) reduces to

$$\omega' = \frac{\epsilon+2}{2}\omega^2 + \frac{\epsilon+2}{2}. \tag{11}$$

This is the Riccati equation for the motion of a particle in a central power law potential assuming $E=0$. It is worth noting that no information about the coupling constant k enters the Riccati equation (11). Essentially what we have shown is that any solution of Eq. (4) will also satisfy Eq. (11). The inverse is not necessarily true and should be examined in detail. Indeed, the coupling constant k should be explicitly contained in the solution for $r(\theta)$ (see below).

A special case that deserves to be briefly mentioned is $\epsilon = -2$. With this exponent, the choice $E=0$ is, in general, only possible if $(l^2/2m)+k<0$. Then directly from Eq. (4) we conclude that r'/r is a constant that, of course, is compatible with the Riccati equation (11). However, this constant cannot be determined by means of Eq. (11). This feature is also inherent in the general case.

To discuss the case $\epsilon \neq 2$, we first solve the Riccati equation (11). The solution can be easily found to be

$$\omega(\theta) = \tan\left(\frac{\epsilon+2}{2}\theta + \frac{\beta}{2}\right) = \frac{\sin[(\epsilon+2)\theta+\beta]}{\cos[(\epsilon+2)\theta+\beta]+1}, \tag{12}$$

where β plays a role of the integration constant. Going back to the definition of ω in Eq. (10) we arrive at a solution for $r(\theta)$

$$r(\theta) = \frac{R}{\{1+\cos[(\epsilon+2)\theta+\beta]\}^{1/(\epsilon+2)}}, \tag{13}$$

where R is a constant. As in the case $\epsilon=-2$ this constant can only be determined by inserting Eq. (11) into Eq. (4). The result is

$$R = \left(\frac{l^2}{m|k|}\right)^{1/(\epsilon+2)}. \tag{14}$$

The last two equations represent then the analytical solution of the posed problem. We obtained this solution by transforming the original problem into a Riccati equation. It might be that the laws of motion in Riccati form are only a curiosity. Given, however, the fact that only a few analytical solutions of the central potential problem are known, it is certainly a useful curiosity. Furthermore, it is not excluded that this way opens more general methods to solve problems in mechanics. In this context, we would like to mention here a yet different connection of the central potential problem with the Ermakov nonlinear differential equation [7]. We refer to the following form of the latter equation [8]

$$q(x)\frac{d^2y}{dx^2} + y(x)\frac{d^2q(x)}{dx^2} = \frac{1}{q^2(x)} f\left(\frac{y}{q}\right), \tag{15}$$

which can be solved by the integrals

$$\int \frac{dx}{q^2(x)} + a = \int \frac{d\left(\frac{x}{q}\right)}{\sqrt{\phi\left(\frac{x}{q}\right)+b}}, \tag{16}$$

where a and b are integration constants and

$$\phi(z) \equiv 2\int f(z)dz. \tag{17}$$

Taking $p=\text{const}=m$ and suitably rescaling the distance r with the mass m, Eq. (15) is essentially identical to Eq. (5). Indeed, in this case the integrals in Eq. (17) give

$$t-t_0 = \frac{1}{m}\int_{r_0}^{r} \frac{dr_1}{\sqrt{2mV(r_1) - \frac{l^2}{r_1} + b}} \tag{18}$$

that, with a proper identification of b, is the same as directly integrating Eq. (5). The interplay between the Ermakov equation and the central potential problems can be a useful tool of studying both problems. We conjecture that certain invariants of the Ermakov equation could be also applied to the central potential problems.

III. COSMOLOGICAL ANALOGY

We want to point out here a beautiful but formal cosmological analogy to the results of the preceding section. We recall that in deriving the Riccati equation (9) we relied on a power law potential (6), a new parameter θ [the angle given in Eq. (9)], and the assumption $E=0$. The analogy to cosmology is based on these observations. In Friedmann-Robertson-Walker space-time the set of Einstein's equations with the cosmological constant Λ set to zero reduce to differential equations for the scale factor $a(t)$, which is a function of the comoving time t. Together with the conservation of energy-momentum tensor they are given by

$$3\ddot{a}(t) = -4\pi G[\rho + 3p(\rho)]a(t), \tag{19}$$

$$a(t)\ddot{a}(t) + 2\dot{a}^2(t) + 2\kappa = 4\pi G[\rho - p(\rho)]a^2(t), \tag{20}$$

$$\dot{p}a^3(t) = \frac{d}{dt}[a^2(\rho + p(\rho)]. \tag{21}$$

In the above G is the Newtonian coupling constant, p is the pressure, ρ is the density, and κ can take the values $0, \pm 1$. Choosing the equation of state to be barotropic,

$$p(\rho) = (\gamma - 1)\rho, \tag{22}$$

essentially fixes ρ to obey a power law behavior of the form

$$\rho = \rho_0\left(\frac{a}{a_0}\right)^{-3\gamma}, \tag{23}$$

and the remaining equations for $a(t)$ reduce to a single equation, viz,

$$\frac{\ddot{a}(t)}{a(t)}+c\left(\frac{\dot{a}(t)}{a(t)}\right)^2+c\frac{\kappa}{a^2(t)}=0, \quad c\equiv\frac{3}{2}c-1. \tag{24}$$

Introducing the conformal time η by

$$\frac{d\eta}{dt}=\frac{1}{a(\eta)}, \tag{25}$$

it can be seen that Eq. (24) is equivalent to a Riccati equation in the function $u=a'/a$, where the dot means derivation with respect to η

$$u'+cu^2+\kappa c=0. \tag{26}$$

This cosmological Riccati equation has been previously obtained by Faraoni [9] and also discussed by Rosu [10] in the context of late cosmological acceleration. The formal analogy to the mechanical case is obvious: the condition $E=0$ corresponds to $\Lambda=0$, the angle θ is replaced by the conformal time η, and whereas in the mechanical example we had a power law behavior of the potential, the barotropic equation of state forces upon ρ to satisfy $\rho\propto a^{-3\gamma}$. As Eq. (11) does not contain the coupling constant k, the cosmological Riccati equation (26) loses the information about G.

IV. QUADRATIC FRICTION

Starting with a constant force g (free fall, constant electric field, etc.) and adding a quadratic friction with a positive friction coefficient $\nu>0$, we have, per excellence, a Riccati equation for the Newton's law of motion

$$\dot{v}=g'-\alpha v^2, \tag{27}$$

with $g'\equiv g/m$ and $\alpha\equiv\nu/m$. The general solution (which for reasons to be seen later in the text we denote by v_p) involves a free parameter λ and reads [11]

$$v_p(t;g',\alpha,\lambda)=\frac{r}{\alpha}\left(\frac{e^{rt}-\lambda e^{-rt}}{e^{rt}+\lambda e^{-rt}}\right), \quad r\equiv\alpha g'. \tag{28}$$

In the following we borrow some techniques from supersymmetric quantum mechanics. However, we do not follow strictly the supersymmetric scheme as the purposes in the quantum case and the mechanical case are quite different. We define a new time-dependent force by

$$\gamma(t;g',\alpha,\lambda,\lambda_1)\equiv\dot{v}_p(t;g',\alpha,\lambda)+\lambda_1 v_p^2(t;g',\alpha,\lambda), \tag{29}$$

with a new parameter $\lambda_1>0$. We emphasize that Eq. (29) is a definition given through the solution Eq. (28). From Eq. (27) it can be then deduced that the following equivalent form of γ can be obtained

$$\gamma=g'-(\alpha-\lambda_1)v_p^2. \tag{30}$$

This resembles supersymmetric quantum mechanics and we might be tempted to compare v_p to Witten's superpotential.

To the new force we again add a quadratic function with a friction coefficient λ_1 such that the new equation of motion becomes

$$\dot{v}=\gamma-\lambda_1 v^2. \tag{31}$$

This has the advantage that per construction v_p is a particular solution of Eq. (31). Equipped with this fact, one can proceed to construct the general solution that is a standard procedure in the general theory of the Riccati equation. Before doing so, it is instructive to dwell upon the physical meaning of the new force γ. Imposing $g'-(\alpha-\lambda_1)(r^2/\alpha^2)>0$, it can be seen that $\gamma>0$. Moreover, as obvious from Eqs. (28) and (30), γ goes to a constant for large t and has a kinklike behavior. We can then envisage a situation where γ is a 'switch-on' function for a force becoming constant at some time. As mentioned above, by construction the problem (31) is solvable because v_p is a particular solution of Eq. (31). By invoking the standard Bernoulli ansatz for the general solution v_g, namely,

$$v_g=v_p+\frac{1}{V}, \tag{32}$$

we arrive at the differential equation (special case of the Bernoulli equation) for V,

$$\dot{V}=2\lambda_1 v_p V+\lambda_1. \tag{33}$$

Writing v_p as

$$v_p=-\frac{1}{\lambda_1}\frac{\dot{W}_p}{W_p}, \tag{34}$$

where

$$W_p=e^{-\lambda_1\int v_p dt}, \tag{35}$$

one is led to the solution for V

$$V=\frac{\lambda_1\int W_p^2 dt+C}{W_p^2}. \tag{36}$$

The general solution is then given by

$$v_g=v_p+\frac{W_p^2}{\lambda_1\int W_p^2 dt+C}. \tag{37}$$

The initial value problem, $v(0)=v_0$, is solved by fixing C through

$$v_0-\frac{r}{\alpha}\left(\frac{1-\lambda}{1+\lambda}\right)=\frac{1}{C}. \tag{38}$$

Up to integrals, the equation of motion (31) is solved. Setting $\lambda = e^{-2\delta}$, we can rewrite Eq. (28) in the more convenient form

$$v_p = \frac{r}{\alpha}\tanh(rt+\delta). \tag{39}$$

Then W_p can be computed explicitly

$$W_p = \frac{1}{[\cosh(rt+\delta)]^{\lambda_1/\alpha}}. \tag{40}$$

It suffices to assume $\lambda_1 = n\alpha$, $n \in N$ leading to integrals of the type $\int \cosh^{-n}(x)dx$, which can be solved in a closed analytical form by recursion formulas. Of course, the procedure outlined here can be generalized by starting with more complicated forces instead of the constant one.

V. CONCLUSION

In this paper we have pointed out the usefulness of the Riccati equation in studying certain mechanical problems. We derived a Riccati equation for a central potential problem of the power law type assuming $E=0$. This led us to an analytical solution of the problem. In a second step, we generalized the system of a constant force plus a quadratic friction to a time-dependent force and friction. We argued that this time-dependent force serves as a ‘switch-on’ function. The problem turned out to be solvable by means of a construction similar to supersymmetric quantum mechanics. As indicated in the text, both applications can be generalized.

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